Beast Academy 2
Chapter 1: Place Value

Sequence: ••• BA1, Chapter 10 Big Numbers ▸ BA2, Chapter 1 This Chapter ▸ BA2, Chapter 2 Comparing ▸ BA2, Chapter 3 Addition •••

Students beginning BA2 must be able to count by 1’s and by 10’s to at least 100, and should know how to read numbers written in digits up to 999.

Overview

In this chapter, we help students understand how our number system works. Young students often see numbers as just a sequence of symbols they can say in order (like the alphabet). In this chapter, we lay the foundations of place value, breaking, and regrouping.

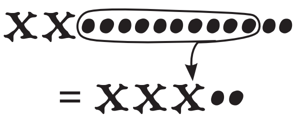
Pirate Numbers

We first introduce a system of “pirate numbers” in which symbols stand for different amounts. A ● stands for one, an X stands for ten, and a C stands for one hundred.

Have students practice writing pirate numbers. Encourage them to find ways to write pirate numbers with as few symbols as possible (by replacing ten ●’s with one X, for example). Students can even practice adding pirate numbers. This gets pretty messy for large numbers and motivates the need for a better system.

Pirate numbers also help students understand breaking and regrouping strategies that we use later for addition and subtraction.

Regrouping with Pirate Numbers



Digits and Place Value

Pirate numbers use ●’s, X’s, and C’s to describe an amount, using up to nine ●’s, X’s, or C’s to write a number.

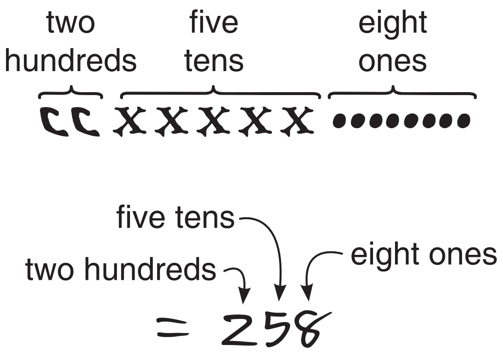
Standard numbers use ten **digits** (0-9) to describe an amount. The location of a digit in a number is called its **place value**. In this chapter, we only include the first three place values: ones, tens, and hundreds.

When writing pirate numbers as standard numbers,

The number of ●’s (ones) goes in the ones place.

The number of X’s (tens) goes in the tens place.

The number of C’s (hundreds) goes in the hundreds place.



Students should see that our standard numbering system is more efficient. Pirate numbers that use ten or twenty symbols can be written with just three digits!

It’s also important to emphasize that *where* each digit is matters. For example, each 5 in 555 stands for something different. Focusing on 0’s can also help students better understand place value. “Why can’t I write ‘4 hundreds and 3 ones’ as 43? How *do* I write a number with 4 hundreds and 3 ones?”

Beast Academy 2

Chapter 1: Place Value

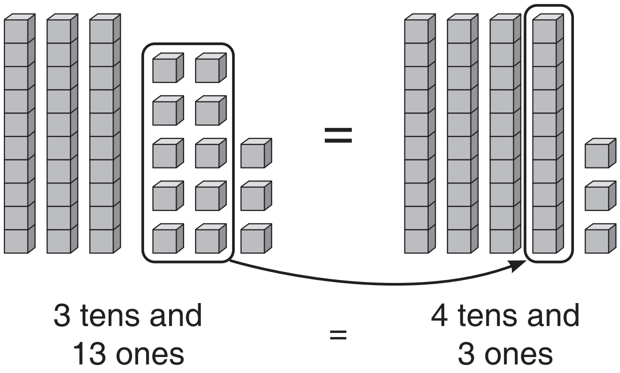
Breaking and Regrouping

Grouping is a natural way to make counting easier.

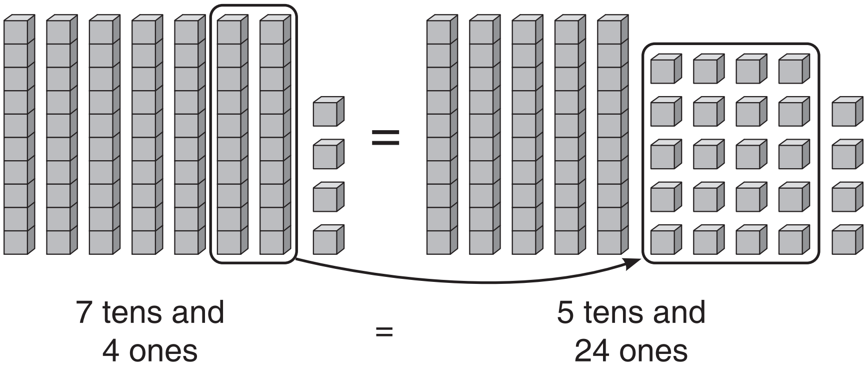
When counting more than a handful of blocks, it makes sense to make groups of ten or one hundred.

Students should be able to show the same value using different groups of tens and ones.

For example, 3 tens and 13 ones is the same as 4 tens and 3 ones. Both equal 43.



Breaking is useful for subtraction. For example, students can see that 7 tens and 4 ones is the same as 5 tens and 24 ones. Manipulatives help make this clear. With practice, students should be able to break and regroup numbers without manipulatives or block diagrams.



It is often helpful for students to recognize 590 as 59 tens. We can show this by breaking the 5 hundreds in 590 into 50 tens. Support this idea with skip-counting.

“If we counts by tens, how many tens does it take to reach 270? 440? 590?” This will help students see the relationship between tens and hundreds.

Adding and Subtracting 1, 10, and 100

Encourage students to describe patterns they find when adding and subtracting 1, 10, or 100. “Adding 1 *usually* increases the ones digit by 1. Adding 10 *usually* increases the tens digit by 1. Adding 100 *usually* increases the hundreds digit by 1. But sometimes we have to regroup.”

Crossing over multiples of 10 or 100 is difficult. Ask students to find strategies that work for them while encouraging them to think about place value.

For example, adding 493+10, we start with 4 hundreds, 9 tens, and 3 ones. Adding one more ten gives us 4 hundreds, 10 tens, and 3 ones. We regroup the 10 tens to make 1 hundred. That gives us 5 hundreds, 0 tens, and 3 ones, which is 503.

$$\begin{aligned} 493 + 10 &= 4\text{ hundreds} + \underline{9\text{ tens}} + 3\text{ ones} + \underline{1\text{ ten}} \\ &= 4\text{ hundreds} + \underline{10\text{ tens}} + 3\text{ ones} \\ &= \underline{5}\text{ hundreds} + 0\text{ tens} + 3\text{ ones} \end{aligned}$$

Students may use other strategies (counting up on your fingers works great in this case), but thinking about place value will help them understand more difficult problems later.

Avoid strategies that involve stacking with “borrowing” and “carrying” that hide some of the place value thinking we want to encourage.

Beast Academy 2
Chapter 2: Comparing

Sequence: ••• BA2, Chapter 1 Place Value ▸ **BA2, Chapter 2 Comparing** ▸ BA2, Chapter 3 Addition ▸ BA2, Chapter 4 Subtraction •••

Students should have a solid understanding of place value for numbers up to 3 digits.

Overview

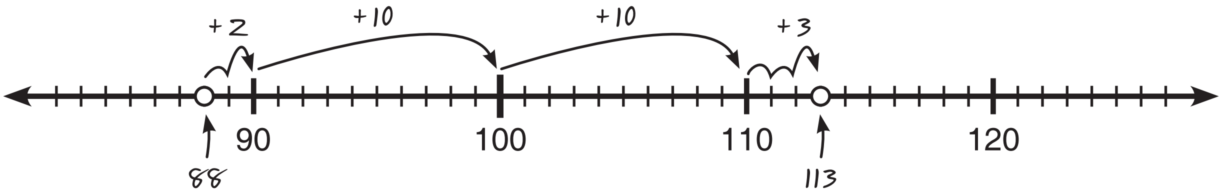
This chapter helps students deepen their understanding of place value and introduces the number line, which is important for students to understand *before* they move on to addition.

The Number Line

The number line is a great tool for visualizing numbers (for now, we focus only on whole numbers). Equally-spaced tick marks show the positions of numbers along the line in order, often with larger tick marks showing the positions of multiples of 10 and 100.

Students should get lots of practice labeling numbers on the number line and finding the distances between them. The distance between any two consecutive numbers (like 89 and 90) is 1 unit. Emphasize that students should be counting one-unit "hops," not the tick marks between numbers. Starting small can help students avoid this common mistake.

Encourage students to find smart ways to find distances: for example, by making hops of 10 instead of counting units one-by-one. These strategies will help them add and subtract later.



From 88 to 113 is 2+10+10+3 = 25 units.

Comparing

An equals sign lets us know that two amounts are the same value.

The < symbol means "is less than". The > symbol means "is greater than". To remember the difference, students often learn that the symbol "eats" the bigger number.

Students should learn to read these symbols both ways. For example, 7<4+4 means "7 is less than 4+4" and "4+4 is greater than 7."

When asked to compare numbers, encourage students to explain their choices using place value. Challenge them to say more than, "It's bigger." Aim for conversations like the ones below.

Q: "How do you *know* that 333 is greater than 88? Aren't 8's bigger than 3's?"
A: "Yes, but 88 doesn't have any hundreds. Any number over 100 is more than a number under 100. So, any number with three digits is more than a number with two digits."

Q: "How do you know that 765 is greater than 576? They have the same digits, and 576 has more tens *and* more ones than 765."
A: "Any number in the 700's is larger than any number in the 500's. So, we only need to look at the hundreds digit to see that 765 is bigger than 576."

Ordering

Once students can compare two numbers, they can apply the same strategies to order groups of numbers.

Discuss the various strategies students come up with on their own and encourage them to explain their reasoning using place value. Avoid giving students a process to apply without understanding why it works.

Encourage all students to stay organized and pay attention to place values.

Order from greatest to least:
88, 78, 788, 887, 877, 77, 878.

It's easier to compare numbers if we align their digits.

We start by stacking the numbers with their digits lined up by place value.

	8	8
	7	8
7	8	8
8	8	7
8	7	7
	7	7
8	7	8

The 3-digit numbers are all bigger than the 2-digit numbers.

The numbers in the 800's are bigger than 788, so we compare them first.

887 has more tens than 877 or 878, so 887 is the biggest, then 878, then 877.

Finally, we order the 2-digit numbers.

8	8	7
8	7	8
8	7	7
7	8	8
	8	8
	7	8
	7	7

If students can reorder a given list of numbers, they are ready to move on to more challenging problems where not all numbers are given. Problems where students have to create the list themselves or fill in missing digits require a solid understanding of place value and good organizational skills.

What are the eight smallest whole numbers
that use only 1's and 9's as digits?

There are only two 1-digit numbers:

1 and 9.

There two 2-digit numbers that start with 1:

11 and 19.

There two 2-digit numbers that start with 9:

91 and 99.

The smallest 3-digit numbers start with 1:

111 and 119.

1, 9, 11, 19, 91, 99, 111, 119

Beast Academy 2
Chapter 3: Addition

Sequence: BA2, Chapter 1 Place Value ▶ BA2, Chapter 2 Comparing ▶ BA2, Chapter 3 This Chapter ▶ BA2, Chapter 4 Subtraction

Students should have a solid understanding of place value for numbers up to 3 digits.

Overview

This chapter introduces students to addition with 2- and 3-digit numbers. Multi-digit addition is often taught as a series of steps where students find single-digit sums, "carry" 1's, and can ignore place value.

We don't recommend teaching any algorithm that students can use without understanding. Instead, focus on using place value and mental computation strategies.

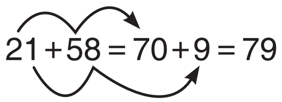
Sums

The result of addition is called a sum. We start by teaching students to add using place value.

Avoid stacking algorithms. While a stacking algorithm might make addition easier for some students, it also lets them ignore place value. For example, when stacking and adding in a problem like 21+58, students can easily forget that they are actually adding 20+50 (2 tens+5 tens), not 2+5.

Instead, encourage students to consider what the digits stand for when adding. We recommend adding the larger place values first since they matter most (when counting change, you start with the quarters). When we add 21+58, we start with 20+50 to get 70, then add 1+8=9 to get 79.

Add 21+58.

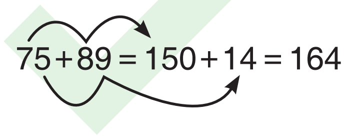


Problems that require **regrouping** are more difficult. Practice adding multiples of ten. Students who can quickly recognize 15 tens as 150 (and vice versa) will have a much easier time regrouping tens and hundreds. In the example on the left, the student adds 70+80 as 7+8 tens, which is 15 tens, or 150. Then, they add the ones.

Add 75+89.

Encourage This

"I know 70+80 is 150.
Adding the ones, I get 5+9=14.
So, the total is 150+14=164."



Not This

"I stack the two numbers.
Add the digits on the right: 5+9=14.
I write 4 below the line and carry the 1.
Add the digits on the left: 1+7+8=16.
I write that below the line. So, I get 164."

$$\begin{array}{r} 75 \\ +\ 89 \\ \hline \end{array} \qquad \begin{array}{r} 1 \\ 75 \\ +\ 89 \\ \hline 4 \end{array} \qquad \begin{array}{r} 1 \\ 75 \\ +\ 89 \\ \hline 164 \end{array}$$

The reasoning on the left helps students develop number sense and improves their estimation and mental computation skills. The steps on the right turn the problem into a series of 1-digit sums that let students get the correct result without understanding the math behind what they've done.

Beast Academy 2

Chapter 3: Addition

Strategies

Using place value is just one of many ways students can learn to add.

Counting Up

To add two numbers, students can start with one and count up by the other. This is especially useful when one of the numbers is close to a number that is easy to add, like 100.

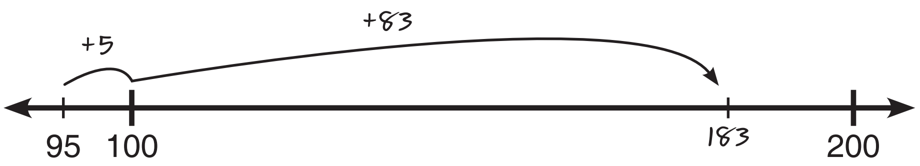
The number line is a great tool to help students visualize this strategy.

“I can start by adding 5 to 95
to get to 100.

To add a total of 88, I need
to add 83 more.

That takes me to 183.
So, 95 + 88 = 183.”

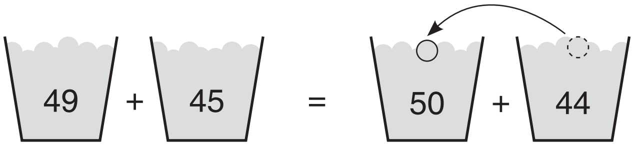
Add 95 + 88.



Compensation

A concrete model helps students understand this strategy. To add two numbers, we can make the addition easier by increasing one number by the same amount that we decrease the other number.

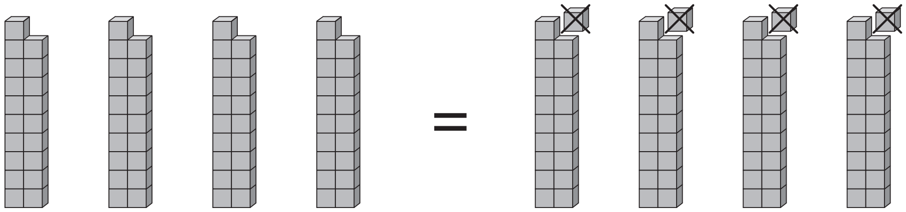
Demonstrate this with students with two buckets of marbles or other items. Moving marbles from one bucket to the other doesn’t change the total number of marbles, but it can make the addition easier.



Adding then taking away

Sometimes it’s easier to add a little extra and then take it away at the end. For example, to add 19 + 19 + 19 + 19, students can add 20 + 20 + 20 + 20 and then take away the four extra.

Use concrete examples and manipulatives to help students understand this strategy.



19 + 19 + 19 + 19 = 20 + 20 + 20 + 20 − 4

Rearranging

You can add numbers in any order. For example, if you are counting all of the marbles in three buckets, it doesn’t matter which two buckets you add first. Pair numbers that make 10’s and 100’s.

$$\begin{aligned} & 18+\underbrace{25+75} \\ = \ & \underbrace{18+100} \\ = \ & 118 \end{aligned}$$

Beast Academy 2

Chapter 4: Subtraction

Sequence:
BA2, Chapter 2 Place Value
BA2, Chapter 3 Addition
BA2, Chapter 4 This Chapter
BA2, Chapter 5 Expressions

Students should have a solid understanding of place value and addition before beginning.

Overview

This chapter introduces students to subtraction. We explain two models of subtraction and help students understand the relationship between addition and subtraction.

Similar to our approach to addition, we purposefully avoid stacking algorithms and instead focus on developing an understanding of place value and mental computation strategies.

Models

There are two important and distinct ways for students to think about subtraction:

- **Taking Away** is how most young students are introduced to subtraction. “If there are 7 apples and I eat 2, how many apples will be left over?”
- **Finding a Difference** is another way for students to think about subtraction by asking how much bigger one number is than another.
  “If Clarence has 7 apples and Pat has 2, how many more apples does Clarence have than Pat?” We use the number line to explore several find-the-difference subtraction strategies.

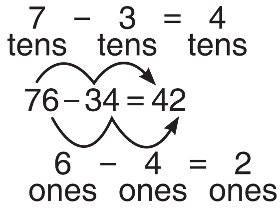
Place Value

Place value subtraction strategies use the take-away model of subtraction. To subtract one number from another, we take away from each place value separately.

Start with problems that do not require breaking like the example on the left.

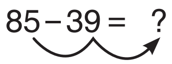
Then, introduce students to breaking as shown in the example on the right.

Subtract 76 − 34.



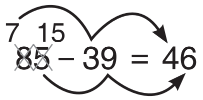
Subtract 85 − 39.

I can’t take 9 ones from 5 ones.
So, I break a ten in 85 into ten ones.



8 tens and 5 ones is the same as 7 tens and 15 ones.

7 tens − 3 tens = 4 tens.
15 ones − 9 ones = 6 ones.
So, 85 − 39 = 46.



We recommend solving these problems without stacking the subtraction to encourage number sense and an understanding of place value. Students who subtract using the stacking algorithm often apply steps without thinking about place value.

Addition & Subtraction

Every subtraction problem can be approached as finding the missing number in a sum. For example, to answer $103 - 95 = \square$, it may be easier to consider $95 + \square = 103$, or, “What number do we add to 95 to get 103?” This leads students to the find-the-difference model of subtraction.

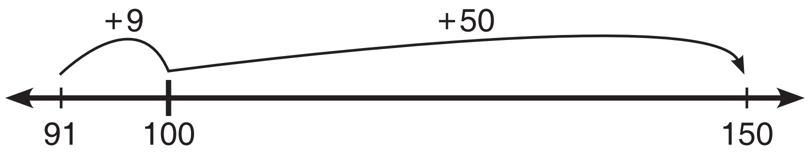
Beast Academy 2
Chapter 4: Subtraction

Strategies

Counting Up

Here, we rely on the find-the-difference model of subtraction. Subtraction tells us how much bigger one number is than another. So, to subtract 150 − 91, we can count up from 91 to 150. To get from 91 to 150, we add 59. So, 150 − 91 = 59.

Subtract 150 − 91.

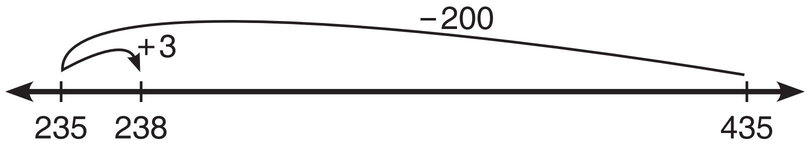


150 is 50 + 9 = 59 more than 91. So, 150 − 91 = 59.

Subtract then Add

Sometimes, it's easiest to take away a little extra, then add the extra back. For example, to subtract 197 from 435, we can take away 200 then add back 3. So, 435 − 197 = 435 − 200 + 3 = 238.

Subtract 435 − 197.

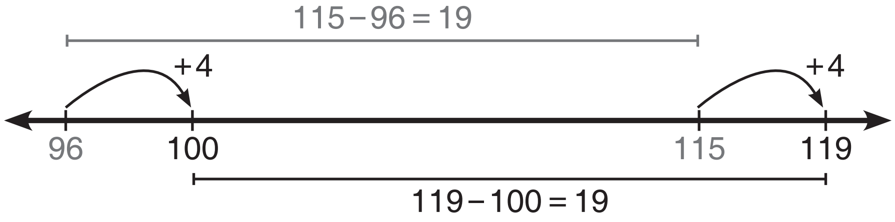


435 − 200 = 235. So, 435 − 197 is 235 + 3 = 238.

Changing a Difference

Subtracting two numbers gives their difference (the distance between them on the number line). We can shift a difference up or down the number line to make it easier to compute. To subtract 115 − 96, we can add 4 to both numbers to make the subtraction easier. 119 − 100 = 19, so 115 − 96 = 19.

Subtract 115 − 96.

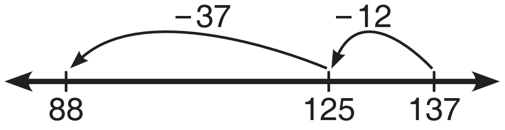


115 − 96 equals 119 − 100 = 19.

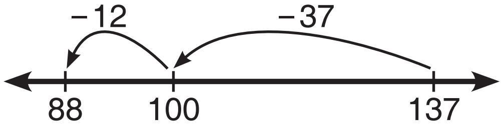
Ordering Strategies

If we start with an amount and subtract several numbers, we can subtract the numbers in any order. Concrete examples help students understand why this works. "Starting with 137 jelly beans, does eating 12 then giving away 37 give the same result as giving away 37 then eating 12?" It does!

Why is this useful? To compute 137 − 12 − 37, it's much easier to subtract 37 first.



137 − 12 − 37 is equal to 137 − 37 − 12



Similarly, to subtract 137 − 49, we can subtract 49 in parts. First, we subtract 37 to get to 100. Then, we subtract the remaining 12 to get 88. So, 137 − 49 = 137 − 37 − 12 = 88.

In other problems, it's best to take away everything at once. To subtract 227 − 44 − 56, we are subtracting a total of 44 + 56 = 100. So, 227 − 44 − 56 is the same as 227 − 100, which is 127.

Beast Academy 2
Chapter 5: Expressions

Sequence: BA2, Chapter 3 Addition ▸ BA2, Chapter 4 Subtraction ▸ BA2, Chapter 5 This Chapter ▸ BA2, Chapter 8 Strategies (+ & −) •••

Students should understand the addition and subtraction strategies in chapters 3 and 4.

Overview

This chapter covers a lot of ground. We learn to evaluate expressions, we begin using parentheses, we use symbols like ⬢ and ☾ to stand for unknown numbers (introducing students to variables), we learn to simplify expressions, and we solve basic equations. We also learn to write math expressions to solve word problems and to represent situations.

This gives an early introduction to math that students will see in prealgebra and should make working with more complex variables and expressions later on less intimidating.

Expressions

An **expression** uses numbers and operations (like + and −) to stand for a value. For example, 20−5+7 and 9+7−6 are both math expressions. (A number on its own like 8 is also an expression.)

To **evaluate** an expression means to find its value. We can evaluate an addition and subtraction expression by working from left to right. So, 9+7−6 means, “Add 9 plus 7, then subtract 6.” 20−5+7 means, “Start with 20, subtract 5, then add 7,” or, “Take 5 from 20 then add 7.”

Have students practice explaining what expressions mean in words and write their own problems. Then, ask students to write expressions to represent statements and word problems.

Write expressions for the statements below.

The sum of 5 and 9.	4 less than the sum of 5 and 9.	Ronnie has 10 pencils. He buys 4 more, then loses 7.	Lizzie has a box of 30 books. She donates 8 dragon books and 5 coloring books to the library.
5+9 or 9+5	5+9−4 or 9+5−4	10+4−7	30−8−5 or 30−(8+5)

In the fourth example, we use **parentheses**. In an expression, what’s in parentheses gets done first. So, for 30−(8+5), we start by adding 8+5 to get 13. Then, we subtract 30−13 = 17.

If there is more than 1 pair of parentheses in an expression, evaluate the inner parentheses first. Students should practice evaluating expressions. Once they are proficient, they can write their own expressions using parentheses and explore how using parentheses can change an expression.

Evaluate 10−(7−(1+4)).

$$\begin{aligned} &10-(7-(\underbrace{1+4})) \\ =\ &10-(\underbrace{7-5}) \\ =\ &\underbrace{10-2} \\ =\ &8 \end{aligned}$$

Place one pair of parentheses in 8−4+2−1 to make an expression that equals 1.

8−(4+2−1) = 3 ✗

8−4+(2−1) = 5 ✗

8−(4+2)−1 = 1 ✓

This works!

Beast Academy 2

Chapter 5: Expressions

Symbols (An Intro to Variables)

Variables are a useful tool to help students explain relationships and patterns. Introducing variable early helps keep students from being intimidated when they see variables later in prealgebra.

In this chapter, we use basic shapes like ★ and ◆ to stand for unknown amounts. In an expression, a symbol like ◆ stands for the same number every time it appears.

Students can think of these symbols as blanks or empty boxes to be filled in. Have them practice evaluating expressions by replacing symbols with different values.

Evaluate 11+(10−★) for each value of ★ below.

★ = 2	★ = 6	★ = 10
11+(10−★)	11+(10−★)	11+(10−★)
= 11+(10−$\underline{2}$)	= 11+(10−$\underline{6}$)	= 11+(10−$\underline{10}$)
= 11+8	= 11+4	= 11+0
= 19	= 15	= 11

Help students understand how expressions that include symbols could be useful. In the Guide, R&G use symbols to stand for values that change. For example, ■ stands for the day of the month. Each day, R&G plan to do (■+10) push-ups.

Encourage students to write expressions that use symbols as variables. For example, if ✤ stands for Xavier's age today, ask students to write an expression for Xavier's age next year (✤+1), or last year (✤−1), or for the age of his big brother Sam, who is 5 years older than Xavier (✤+5).

Simplifying

Simplifying an expression means writing it in a way that is easier to understand. For expressions that only use numbers, this is the same as evaluating the expression. For example, 8+5 simplifies to 13.

We can also simplify expressions that use symbols. For example, no matter what value we use for ◘, ◘−◘ is always 0, so we say that ◘−◘ simplifies to 0. Similarly, 5+◘−◘ simplifies to 5+0, or 5.

Simplify 💧−5+5.

Subtracting 5 then adding 5 is the same as doing nothing, so 💧−5+5 simplifies to 💧.

Simplify (✹+✹)−(✹+✹)+✹.

Subtracting (✹+✹) from (✹+✹) gives us 0. So, (✹+✹)−(✹+✹)+✹ is 0+✹, which is just ✹.

Equations

An equation shows that two expressions are equal (equations have equals signs, expressions don't).

Solving an equation that includes a symbol means finding the value of the symbol. Students can think of this as filling in a blank. Solving ★+★−5=21 is the same as filling both blanks in __+__−5=21 with the same number. Students can guess and check to find these answers. (Here, ★=13.)

Beast Academy 2

Chapter 6: Problem Solving

Sequence: BA1, Chapter 12 Problem Solving ▶ BA2, Chapter 6 This Chapter ▶ BA2, Chapter 9 Odds & Evens ▶ BA2, Chapter 12 Problem Solving

This is a tough chapter that introduces some useful skills for solving unfamiliar problems. The strategies in this chapter require very few computational skills. They can be taught at any time and used throughout the Beast Academy curriculum.

Overview

The math in Beast Academy is primarily a backdrop for teaching students how to solve problems.

The goal is to give students ways to approach unfamiliar problems.

If ▲ + ★ + ★ = 17 and ▲ + ▲ + ★ = 13, what is ▲ + ★?

Solving hard problems can be frustrating for students who are used to having a clear path of worked examples and step-by-step instructions. It is important to encourage failure as a vital part of learning.

Students who always get the right answer on their first try probably aren't learning efficiently, and can develop perfectionist tendencies. Encouraging a variety of strategies for solving unfamiliar problems is a great way to combat perfectionism and build patience and resilience.

"Try guessing values of ▲ above. If ▲ is 7 in the first equation, what is ★? Do those values work for the second equation? No? That's ok, what should we guess next?"

Teaching students to play, experiment, fiddle, fail, and repeat helps them become resourceful, resilient problem solvers who will be better at math and everything else they do.

Guessing and Checking

Many students don't like guessing, and many teachers discourage it.

Make sure students know that the goal is to learn more about the problem, not to get the right answer on the first try. We don't care whether our first guess is wrong as long as we learn something from it. Brilliant adults in every profession guess all the time.

Model guessing for students, even bad guessing! Below, we are trying to find the value of ☾:

☾ + ☾ + 5 = 33

"Let's try ☾ = 100. Is that a good guess?"

"Noooo!!!"

"Why is 100 a bad guess?"

"It's too biiiig! It's even *more* than 33!"

"What should we guess instead?"

Encourage first guesses that are easy to compute with. In that sense, 100 really wasn't a bad guess, and we eliminated a lot of numbers! Talk about what each guess teaches us.

Guessing and checking is a great way for students to build number sense. It also encourages other great problem solving strategies like staying organized by keeping track of guesses.

Beast Academy 2
Chapter 6: Problem Solving

Working Backwards

Some problems are best solved by starting at the end and working backwards. Like the guess-and-check problems before, these problems require students to stay organized. Here's a tough example:

> Rashaad does push-ups every night for 3 nights in a row. Each night, he does 1 less than double the number of push-ups he did the night before. On the third night, he does 41 push-ups! How many did Rashaad do on the first night?

Students may be tempted to use their new guess-and-check skills. Encourage a work-backwards strategy instead. Ask students to figure out how many push-ups Rashaad did on the second night. Students should come up with something like this:

> "We know 41 is 1 *less* than double the push-ups Rashaad did on night 2.
> That means $41+1=42$ is double the number of push-ups Rashaad did on night 2.
> $21+21=42$, so Rashaad did 21 push-ups on night 2."

Ask students to explain their reasoning and help them stay organized. Students can use the same reasoning used above to see that Rashaad did 11 push-ups on night 1.

Once students have their answer, ask them to check their work.

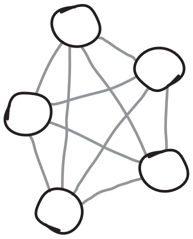
$$\begin{tikzcd}
\underline{\;11\;} \arrow[r, bend left, "\text{Double, then -1}"] & \underline{\;21\;} \arrow[r, bend left, "\text{Double, then -1}"] & \underline{\;41\;}\ \checkmark
\end{tikzcd}$$

Draw a Picture

Drawing a simple sketch can help students organize and make sense of all the information in a problem. Make sure students know that the drawings are just for modeling the problem and not meant to be works of art.

For example, in the diagram on the right, each circle is a monster and each line connecting two monsters stands for a handshake.

> Five monsters all shake hands with each other before a game. How many hand shakes are there?



It takes 10 gray lines to connect all five monsters to every other monster, so there are a total of 10 handshakes.

Extra Information

Most real problems don't come with all of the information neatly organized.

It is important for students to be able to find the necessary information in a problem and ignore any useless extra information. Encourage students to carefully read problems, figure out what the problem is asking, and find the useful information required to solve it.

Beast Academy 2
Chapter 7: Measurement

Sequence: BA1, Chapter 11 Measurement → BA2, Chapter 7 This Chapter → BA3, Chapter 9 Measurement

Students should be able to add and subtract fluently before beginning this chapter.

Overview

This chapter only includes length measurements (not weight, volume, time, etc.) We use standard units like inches and centimeters to measure lengths.

Encourage students to understand and use units correctly.

“How long is the pencil?” “Five.” “Five *what*? Years? Gallons?”

Measurements need units to be meaningful. However, we don’t recommend marking answers *wrong* when students forget to include units.

Units

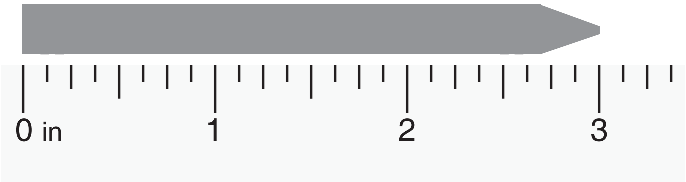
To compare two lengths, measure both using the same **unit**. For example, you could measure the length of two different desks using crayons, placing them end-to-end. Model some possible pitfalls of this method and discuss them with students. For example, “Why can’t I use the short crayons to measure one desk, and the long crayons to measure the other?”

Have students use other items as units of measure, and discuss which ones work best (generally ones that do not vary in size). Introduce standard units of length starting with inches and centimeters, which can be found on most school rulers (in the United States).

Using a Ruler

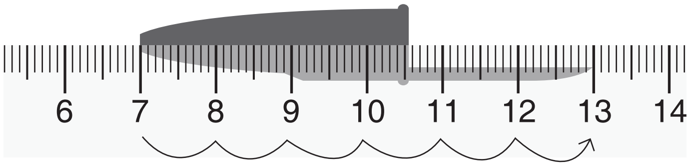
Measuring with a ruler can be tricky. Demonstrate how to use a ruler to measure objects in inches and in centimeters. Place the mark labeled “0” at one end of the object and read the length on the ruler at the other end of the object.

This crayon is **3 inches** long.



Discuss lengths of objects that are measured without placing one end at 0. For example, “How long is a pen cap if one end is at the 7 cm mark and the other end is at the 13 cm mark?” The difference between these values gives the correct length of the cap (6 cm), but it’s easier to just put one end at 0. Students can also count the number of 1 cm “hops” it takes to get from 7 to 13.

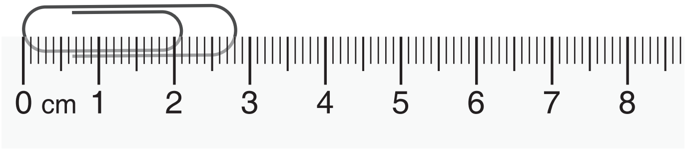
This pen cap is 13 − 7 = **6 centimeters** long.



Six 1 cm hops = **6 centimeters**.

Begin with whole-number lengths in inches or centimeters. Students should then measure objects to the nearest whole unit. “Is the paperclip closest to 2 cm, 3 cm, or 4 cm long?”

This paperclip is *about* **3 centimeters** long.



Beast Academy 2
Chapter 7: Measurement

Changing Units

We introduce other units of measure including customary units of length (feet, yards, miles) and metric units (meters and kilometers) that can be used to measure greater lengths. The goal is to give students a sense of the size of these larger units and their usefulness. Students do not need to memorize conversion factors, but can make their own notes like the ones below to use as needed.

Customary units of length:

1 inch (the length of a small paperclip)
1 foot = 12 inches (the length of a big shoe)
1 yard = 3 feet = 36 inches (the width of a school door)

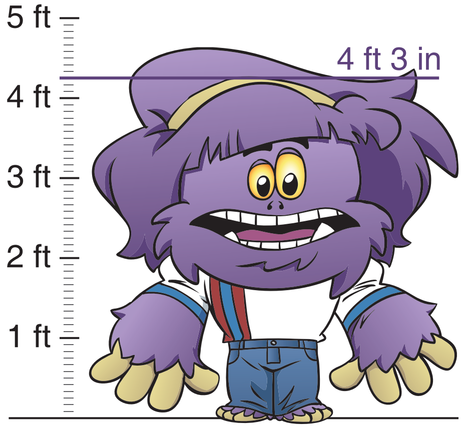
It is typically easier for students to change from large units to small units than the other way around, since the reasoning is similar to other problems they've seen. "How many inches are in 4 feet?" is similar to asking, "There are 12 apples in each of 4 baskets, how many apples are there all together?" Students add 12+12+12+12 to answer both.

When changing from a small unit to a larger one, students do not need to use division. Keep the numbers small and encourage a variety of strategies. For example, to answer "How many yards are there in 18 feet?" students can make a chart like the one below.

Feet	3 feet	6 feet	9 feet	12 feet	15 feet	18 feet
Yards	1 yard	2 yards	3 yards	4 yards	5 yards	6 yards

Mixed Measures

Mixing feet and inches is common in customary measurements. For example, many students may know their height in feet and inches. This is a good opportunity for some hands-on measurement. Remind students that the number of inches in a mixed measure must always be less than 12, and encourage regrouping (3 ft 15 in = 4 ft 3 in).



Addition and Subtraction

This is the first time we stack addition or subtraction in Beast Academy. Students line up the feet and inches and use the breaking and regrouping strategies we used with multi-digit addition and subtraction.

Add 3 ft 11 in + 2 ft 8 in

$$\begin{array}{rrr} & 3\text{ ft} & 11\text{ in} \\ + & 2\text{ ft} & 8\text{ in} \\ \hline & \cancel{5\text{ ft}} & \cancel{19\text{ in}} \\ & 6\text{ ft} & 7\text{ in} \end{array}$$

Subtract 4 ft 3 in − 1 ft 7 in

$$\begin{array}{rrr} & 3\text{ ft} & 15\text{ in} \\ & \cancel{4\text{ ft}} & \cancel{3\text{ in}} \\ - & 1\text{ ft} & 7\text{ in} \\ \hline & 2\text{ ft} & 8\text{ in} \end{array}$$

Students who have learned to stack addition and subtraction must be careful here. For example, adding 11+8 to get 19 inches, students may be tempted to write 9 and "carry" the 1.

Beast Academy 2

Chapter 8: Strategies (+&−)

Sequence: ••• BA2, Chapter 4 Subtraction ▶ BA2, Chapter 5 Expressions ▶ **BA2, Chapter 8 This Chapter** ▶ BA2, Chapter 10 Big Numbers •••

Students should be comfortable with the previously-taught strategies from BA2 Chapters 3 and 4 before beginning this chapter. These strategies are reviewed in the beginning of the chapter.

Overview

The goal is to help students read and understand math expressions. Students who understand expressions can rearrange, rewrite, and simplify them efficiently. Providing real-world examples helps students make sense of why these strategies work, which will make evaluating many addition and subtraction expressions easier.

Rearranging

We can rearrange the order of addition and subtraction expressions to make computations easier. Concrete examples help students see how, when, and why we can switch the order of addition and subtraction.

"If I begin with 186 apples, then pick 145 more, then sell 86, how many apples will I have?" Students can write an expression and compute from left-to-right to get 186 + 145 − 86 = 245 apples.

"What if I start with 186 apples and sell 86 apples *before* I pick 145 more? Will I end up with the same number of apples as before?" Help students recognize that the result will be the same, and ask them to explain why this may be useful.

Evaluate 186 + 145 − 86.

$$\begin{aligned} &186\,\boxed{+145}\,\boxed{-86}\\ =\;&\underbrace{186-86}+145\\ =\;&\underbrace{100+145}\\ =\;&245 \end{aligned}$$

Adding 145 then subtracting 86 is the same as subtracting 86 then adding 145. In other words, when you rearrange addition and subtraction, always keep the + and − signs "glued" to the numbers you are adding or subtracting.

Important: You can't just rearrange the *numbers*. 186 + 145 − 86 is *not* equal to 186 + 86 − 145.

Canceling and Almost Canceling

Many operations undo each other. For example, adding 8 then taking away 8 is the same as doing nothing. Models help make this clear. "If I find $8 then spend $8, do I have more or less money than I started with?" Students can see that you'll have the same amount. We say that +8 and −8 "cancel" each other.

Evaluate 97 − 8 + 8 − 7 + 7 − 6.

$$\begin{aligned} &97\cancel{-8}\cancel{+8}\cancel{-7}\cancel{+7}-6\\ =\;&97-6\\ =\;&91 \end{aligned}$$

Recognizing operations that *almost* cancel can also be useful. "If I spend $73 then earn $72, do I have more or less money than I started with? How much less?" Apply the same reasoning to expressions.

To compute 158 − 73 + 72, we start with 158 and subtract 1 more than we add. So, 158 − 73 + 72 is equal to 158 − 1 = 157 (which is much easier than computing from left to right).

Evaluate 158 − 73 + 72.

$$\begin{aligned} &158-73+72\\ =\;&158-1\\ =\;&157 \end{aligned}$$

Beast Academy 2

Chapter 8: Strategies (+&−)

All at Once

Sometimes it's easiest to compute the addition all at once and the subtraction all at once.

Real-world examples can help students understand.
"A train with 123 passengers makes two stops. At the first stop, 7 passengers get off the train and 20 passengers get on. At the second stop, 13 passengers get off and 30 get on. How many passengers are on the train after the second stop?"

Evaluate 123 − 7 + 20 − 13 + 30.

$$\begin{aligned} &123 - 7 + 20 - 13 + 30 \\ =\ &123 - 20 + 50 \\ =\ &103 + 50 \\ =\ &\mathbf{153} \end{aligned}$$

Encourage students to consider the subtraction and addition separately.
"All together, how many passengers got off the train? How many passengers got on?"

Students can then apply an all-at-once strategy to evaluate addition and subtraction expressions like the example below, and combine this with other strategies when appropriate.

Evaluate 111 + 45 − 87 + 55 − 10.

Starting with 111, we add a total of 45 + 55 = 100, and subtract a total of 87 + 10 = 97.

$$\begin{aligned} &111 + 45 - 87 + 55 - 10 \\ =\ &111 + 100 - 97 \end{aligned}$$

Adding 100 then subtracting 97, we add 3 more than we subtract. So, it's the same as adding 3.

$$\begin{aligned} =\ &111 + 100 - 97 \\ =\ &111 + 3 \\ =\ &114 \end{aligned}$$

Parentheses

Students should be able to make sense of math expressions that include parentheses. Again, examples can help students understand what expressions that include parentheses mean.

On Tuesday, Melvin sold 45 apples and 16 oranges.
On Wednesday, Melvin sold 55 apples and 17 oranges.
How many more fruits did Melvin sell on Wednesday than Tuesday?

Method 1: Melvin sold 55 + 17 fruits Wednesday and 45 + 16 Tuesday. We subtract these totals.

$$\begin{aligned} &(55+17)-(45+16) \\ =\ &72 - 61 \\ =\ &11 \end{aligned}$$

Method 2: On Wednesday, Melvin sold 55 − 45 more apples and 17 − 16 more oranges than Tuesday. We add these differences.

$$\begin{aligned} &(55-45)+(17-16) \\ =\ &10 + 1 \\ =\ &11 \end{aligned}$$

Ask, "Which method was easier?" Give problems where students decide what expression to use.

Students can then apply similar reasoning to evaluate expressions like (184 + 184) − (183 + 183) quickly. "Since 184 is 1 more than 183, the sum in parentheses on the left is 2 more than the sum on the right. So, (184 + 184) − (183 + 183) = 2."

Skip-Counting

This section gives students a chance to practice repeated addition by skip-counting, which is an important step towards multiplication.

Beast Academy 2
Chapter 9: Odds & Evens

Sequence: BA1, Chapter 12 Problem Solving ▸ BA2, Chapter 6 Problem Solving ▸ BA2, Chapter 9 This Chapter ▸ BA2, Chapter 12 Problem Solving •••

Understanding parity (odds and evens) is useful for solving a large variety of problems, so we've included this chapter in the problem-solving sequence above. However, it can be completed at any point after students are comfortable with multi-digit addition and subtraction.

It's not essential, but it's a lot of fun and teaches some useful problem solving strategies, so we encourage everyone with time to tackle the math in this chapter.

Overview

There is much more to odds and evens than just memorizing that odds end in 1, 3, 5, 7, or 9, and evens end in 2, 4, 6, 8, or 0. In this chapter, we explain the basic properties of odds and evens, then ask students to apply what they know to find clever ways to approach different problem types.

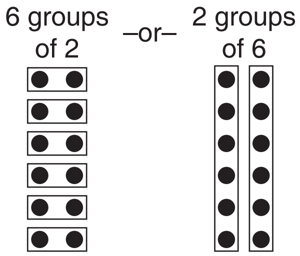
Basics

Give students two ways to think about even numbers:

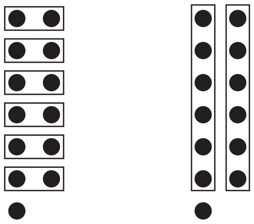
- An even number of items can be split into **groups of two** with no extras.
- An even number of items can be split into **two equal groups** with no extras.

Whole numbers that are not even are called odd. No matter how you try to split an odd number, there will always be one extra.

12 is **even**.
12 can be split into:



13 is **odd**.
13 leaves 1 extra when we try to split it into groups of 2 or 2 equal groups.

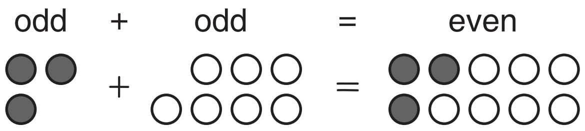


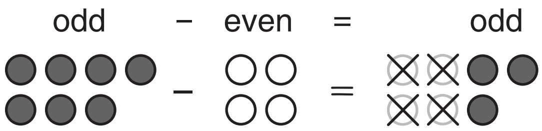
Addition and Subtraction

Have students practice adding and subtracting combinations of odds and evens to look for patterns.

Ask questions like, "Can anyone find two odd numbers that have an odd sum?" and "If I subtract two odd numbers, is the result always even?" Encourage students to use the models above to help confirm and explain the patterns they find.

Students will not need to memorize rules like *odd + odd = even* if they understand how to apply the models above. "When we add two odds, we can pair up the extras to make another two!"





Beast Academy 2

Chapter 9: Odds & Evens

Adding More Than Two Numbers

“What if we add a *thousand* even numbers? Can you tell if the sum is odd or even?”

Encourage students to use the models again. “Since every even number can be split into 2’s, no matter how many evens we add, we end up with a bunch of 2’s, which is even; you always get an even result.”

“What if we add a thousand *odd* numbers? Can you tell if the sum is odd or even?”

Adding odds is trickier. Try adding 3, 4, 5, or 6 odd numbers and look for a pattern (this is a great problem-solving strategy: start with a simpler problem). Students should see that adding an odd number of odds gives an odd result, while adding an even number of odds gives an even result.

“Why is that?”

If there are an even number of odds, we can make pairs of odd numbers that have an even sum.

If there are an odd number of odds, we can make pairs of odd numbers that have an even sum, but there will always be one left out. This leftover odd number makes the whole sum odd.

Problem Solving with Parity (Odds & Evens)

Many classic problems can be solved using odds and evens. Don’t give away the “tricks” here. Most of the fun is in discovering them.

Inside/Outside Problems

The lines on the map to the right trace the jagged islands of Breaker Bay. If A is on land, is B on land or in water?

How can we tell? That map is terrible!

Every time you cross a line, you switch between land and water. For example, if we start at A and cross 1 line, we end up in the water. If we cross 2 lines, we end up back on land. We start to see that crossing an *odd* number of lines always puts us in the water, while an *even* number puts us on land. From A to B, we always cross an even number of lines, so B is on land.

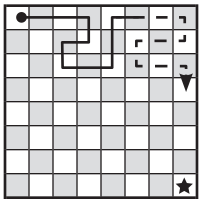
Coin Problems

Suppose you have 9 coins, all on heads. You can flip any two coins at a time. Can you flip all 9 coins to tails? Try it.

It’s impossible! Why?!?

Flipping two coins at once, you can decrease the number of heads by 2 (by flipping HH to TT) or increase the number of heads by 2 (by flipping TT to HH) or keep the number of heads the same (by flipping TH to HT or HT to TH). Since you start with an odd number of heads, you can never get an even number of heads. So, there will always be at least 1 heads. The same is true whether you start with 3 heads, 5 heads, or 99 heads—which seems cruel, but may keep a student busy for a while!

Checkerboards



Can we trace a path from the top-left corner to the bottom-right corner of a standard 8-by-8 checkerboard that passes through every square exactly once? (No diagonal moves allowed.)

We’re out of room! You’ll need to read the Guide to find out.

Beast Academy 2
Chapter 10: Big Numbers

Sequence: ••• BA2, Chapter 5 Expressions ▶ BA2, Chapter 8 Strategies ▶ BA2, Chapter 10 This Chapter ▶ BA2, Chapter 11 Algorithms •••

Students should have a good understanding of place value including comparison, addition, and subtraction strategies, and regrouping before beginning this chapter.

Overview

Students who are fluent with operations involving 3-digit numbers can use the same tools with larger numbers. In this chapter, we help students apply what they've learned to numbers greater than 999.

Larger Place Values

Teach the names of place values in numbers with more than three digits. The digits in these numbers are separated by commas that separate the millions, the thousands, and the rest.

When we read a large number like 263,406,189, we read the millions first, then the thousands, then the rest. So, we read 263,406,189 as "two hundred sixty-three million, four hundred six thousand, one hundred eighty-nine."

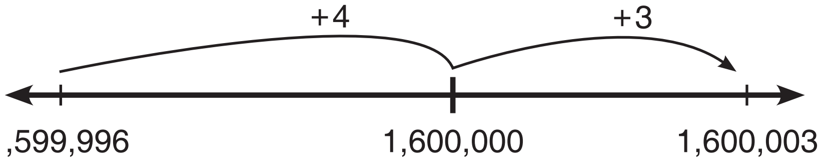
Counting Up

Students may have trouble counting up when digits "roll over," for example, when adding 1 to 1,599,999 to reach 1,600,000. It's useful to have students practice counting up to get used to cases when lots of 9's turn into 0's.

For example, students can solve 1,599,996+7 by counting up:

1,599,996 1,599,997 1,599,998 1,599,999 1,600,000 1,600,001 1,600,002 1,600,003

Next, students can reason that to add 1,599,996+7, they can first add 4 to get to 1,600,000, then add 3 more to get 1,600,003.



1,599,996+7
= 1,600,003

Using familiar strategies like these helps students connect and apply what they already know to much larger numbers. Continue to avoid the standard stacking algorithm (which is a pretty bad way to add 1,599,996+7). See the Chapter 11 overview for more on algorithms.

Beast Academy 2
Chapter 10: Big Numbers

Counting by 10’s, 100’s, 1,000’s

The way we name numbers can make them easier to work with. Counting by 10’s, 100’s, and 1,000’s can help students think of numbers in different ways.

For example, counting by 100’s can help students recognize 1,400 as fourteen hundreds. A student who can recognize 1,400 as “fourteen hundred” will have an easier time adding 600+800, or subtracting 2,100−700.

600+800 is 6 hundreds+8 hundreds
= 14 hundreds, or 1,400.

2,100−700 is 21 hundreds−7 hundreds
= 14 hundreds, or 1,400.

Addition and Subtraction

Adding hundreds, thousands, or millions works the same way as adding anything else. Students who can add 123 apples plus 50 apples can add thousands the same way:
123 thousands plus 50 thousands is 123+50 = 173 thousands, or 173,000.

If a student can add two three-digit numbers, they can add larger numbers. For example, to add 215,354+198,228, we can start by adding the thousands: 215+198 = 413 thousands. Then, we add the rest: 354+228 = 582.

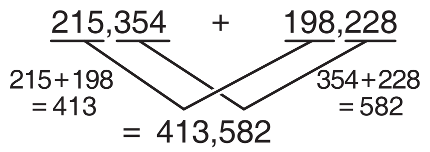
All together, we have 413,582.

Adding the thousands separately helps students focus on place value, and will strengthen their number sense before learning to apply the traditional algorithm in the next chapter.

123 apples + 50 apples = 173 apples

123 thousands + 50 thousands = 173 thousands

123,000 + 50,000 = 173,000



Comparing and Ordering

When comparing numbers, ***where*** the digits are is more important than ***what*** the digits are.

For example, even though 7 is larger than 3, students should recognize that 700,000 is less than 3,000,000. Ask students how they know. It helps to recognize that 3,000,000 is 3,000 thousands.

Encourage students to look for patterns that help them discover some strategies on their own:

- When comparing whole numbers, the one with more digits is always larger.
- When comparing two numbers that have the same number of digits, we look at their largest place values first. For example, 7,333,333 is larger than 6,888,888 because it has more *millions* (even though 6,888,888 has a larger digit in every other place value).

Estimation

Estimation is a practical skill that will help students build number sense. When we compute with large numbers, an exact answer is often not needed. If we need the exact sum of 128,914,548+42,238,936, we use a calculator. If not, we estimate.

A good estimate is **easy to compute and close to the exact amount**. The amount we’re off by should be small compared to the actual value.

Beast Academy 2

Chapter 11: Algorithms (+&−)

Sequence: • • • BA2, Chapter 5 Expressions ▶ BA2, Chapter 8 Strategies ▶ BA2, Chapter 10 Big Numbers ▶ BA2, Chapter 11 This Chapter

Students should be able to use a wide variety of addition and subtraction strategies and have strong mental math skills before learning the addition and subtraction algorithms in this chapter.

Overview

An algorithm is a step-by-step process used to do a specific task, like a recipe. We intentionally avoid teaching traditional algorithms until late in BA2. Students who learn algorithms too early may:

- Lack the tools to understand why the algorithms work.
- Misunderstand and mis-apply steps that have no meaning to them.
- View math as confusing sets of processes and formulas to memorize.
- Resist learning new methods for something they “already know how to do.”
- Develop a “Just show me how” mentality towards math.

By saving the algorithms for last, we give students a chance to learn a variety of strategies for addition and subtraction, helping them build strong number sense and the tools needed to understand the meaning behind the steps in every algorithm we teach.

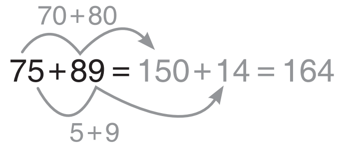
If taught well, algorithms can make some computations more efficient and deepen student understanding of place value.

Encourage students to avoid algorithms when they are unnecessary or counterproductive. For example, give problems like 9,997+5 and 10,003−7 that are best done mentally, where the algorithms involve so much breaking and regrouping that they can lead to some pretty entertaining mistakes.

Stacking Addition

The traditional stacking algorithm is an extension of what we’ve done. Here’s how to tie it to the method we’ve used before.

Until this point, we’ve been writing our addition horizontally in BA. We add the tens first (70+80=150), then the ones (5+9=14), then add the partial sums to get the final sum (150+14=164).



We can stack the addition and add as shown on the right. We’re doing the same steps, just organized differently. Here, we add the ones first (5+9=14) then the tens (70+80=150). These steps introduce students to stacking by place value and help students transition to the traditional algorithm below.

$$\begin{array}{r} 75 \\ +\ 89 \\ \hline 14 \end{array} \qquad \begin{array}{r} 75 \\ +\ 89 \\ \hline 14 \\ +\ 150 \\ \hline \end{array} \qquad \begin{array}{r} 75 \\ +\ 89 \\ \hline 14 \\ +\ 150 \\ \hline 164 \end{array}$$

In the traditional algorithm, we compute each digit of the sum from right to left. In the ones column, 5+9=14. The ones digit of the sum is 4. We regroup or “carry” the ten from 14 to the tens column. Then, in the tens column, 1+7+8=16 tens, or 1 hundred and 6 tens. So, 75+89=164.

$$\begin{array}{r} 75 \\ +\ 89 \\ \hline \end{array} \qquad \begin{array}{r} 1 \\ 75 \\ +\ 89 \\ \hline 4 \end{array} \qquad \begin{array}{r} 1 \\ 75 \\ +\ 89 \\ \hline 164 \end{array}$$

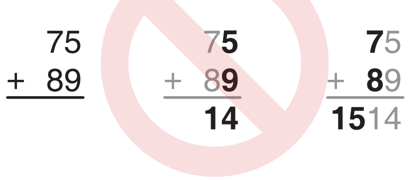
Beast Academy 2
Chapter 11: Algorithms (+&−)

Stacking Addition (Continued)

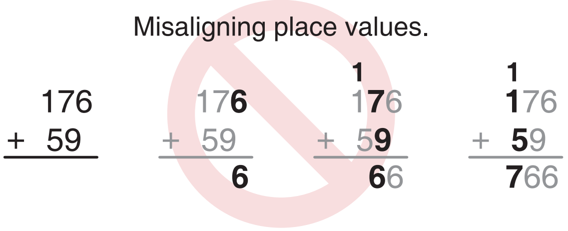
Students who already have a solid understanding of place value and addition should be able to make sense of the steps in the traditional algorithm. Most have probably learned it from a parent or teacher at some point.

However, there are a lot of mistakes students can make when simply following a procedure.

Forgetting to regroup.

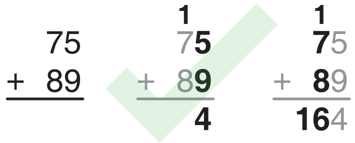


Misaligning place values.



Ask students to narrate what they are doing in terms of place value and state their final answer.

> *"I line up the ones with the ones and the tens with the tens. First, I add the ones to get 14. That's 1 ten and 4 ones, so I write a 4 in the ones place and add 1 ten to the tens. Then, I add the tens: 1+7+8 = 16 tens, which is 1 hundred and 6 tens. So, 75+89 is 164."*



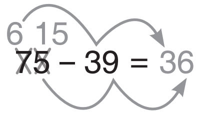
A student with good number sense who says "75+89 is 1,514" aloud is likely to recognize they've done something wrong. A student who has learned to add primarily using the algorithm will have a lot more trouble noticing mistakes.

We scaffold the early addition problems by setting them up at first, then by giving boxes for students to align their addition, and finally asking them to set up and solve addition problems on their own.

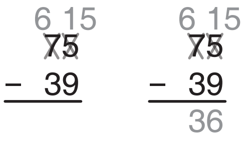
Once students get comfortable adding two numbers, they can move on to adding three or more.

Stacking Subtraction

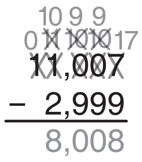
Here's how to tie the method we've used before to the traditional stacking algorithm for subtraction.



Until this point, we've been subtracting horizontally. We start by subtracting the ones. We can't take 9 ones from 5 ones, so we break a ten from 75 to make ten ones. 75 is the same as 6 tens and 15 ones. Then, we subtract by place value. 6 tens minus 3 tens is 3 tens, and 15 ones minus 9 ones is 6 ones.



We can stack and subtract using exactly the same steps. We break a ten from 75 to get 10 ones, writing 75 as 6 tens and 15 ones. Then, we subtract by place value as we did above.



Breaking more than one place value can be a confusing and frustrating process. Encourage students to stack only when needed. For example, students should be able to find a better way to solve 11,007−2,999 than with the messy process shown on the right (probably by subtracting 3,000 and adding 1).

Beast Academy 2

Chapter 12: Problem Solving

Sequence:
BA1, Chapter 12 Problem Solving ▸
BA2, Chapter 6 Problem Solving ▸
BA2, Chapter 9 Odds & Evens ▸
BA2, Chapter 12 Problem Solving •••

Like Chapter 6, this is a tough chapter that introduces some useful skills for solving unfamiliar problems. The strategies in this chapter require very few computational skills. They can be taught at any time and used throughout the Beast Academy curriculum.

Overview

The math in Beast Academy is primarily a backdrop for teaching students how to solve problems.

Problem solving is what we do when we don't know what to do. Sometimes, the hardest part about tackling a problem is getting started. Students should always try the following strategy, which is not mentioned directly in either of our problem solving chapters:

Do something!

Strategies like organizing information, looking for patterns, solving simpler problems, and creating models are all great ways to get started when we're stuck. They can help us overcome the tendency to sit and stare, hoping for the answer to come.

Organization

Organization is not a strength of most young students. So, there's plenty of room for improvement! Organization takes practice. The counting problems in this section are usually very difficult if you don't stay organized.

How many ways can 5 lollipops be shared between
Alice, Ben, and Carrie if each must get at least one?

Encourage students to try to find a strategy that helps them find all of the possibilities. Discuss which work best, which are not as useful, and why. Below are a few ways to organize, but your students may find others.

1: We can split the lollipops two ways: 3+1+1 or 2+2+1. Then, decide who gets how many:

A	B	C
3	1	1
1	**3**	1
1	1	**3**
1	2	2
2	**1**	2
2	2	**1**

6 total ways.

2: Prioritize giving lollipops to Alice, then Ben, then Carrie:

A	B	C
3	1	1
2	2	1
2	1	2
1	3	1
1	2	2
1	1	3

6 total ways.

3: Ignore three lollipops (since everyone gets 1). Draw dots for the other two (using strategy 3):

A	B	C
••	-	-
•	•	-
•	-	•
-	••	-
-	•	•
-	-	••

6 total ways.

Beast Academy 2

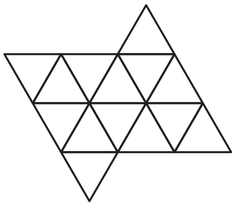
Chapter 12: Problem Solving

Organization (continued)

What makes a good organizational strategy? If you've organized your work well, you can tell when you're finished counting, and you can look back and tell whether you've missed anything.

This is difficult for some problems. For example, when counting triangles in the diagram on the right, it's hard to know for sure if you've missed one (or counted one twice). But, keeping track of small △, medium △, and large △ triangles makes it much easier (the answer is at the end of the page).

How many triangles of any size can you count below?



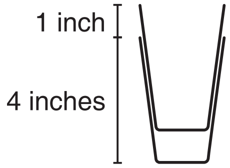
Solve a Simpler Problem

Many problems that seem complex can be solved by trying simpler variations and looking for a pattern. This becomes more useful as students learn to recognize more patterns, but it is a good skill to begin practicing early.

For the problem on the right, encourage students to start with fewer than 20 cups. How tall is 1 cup? How tall is the stack of 2 cups? How tall would a stack of 3 cups be? Is there a pattern? Does the pattern work for 4 cups?

Students should describe and test patterns they find. Challenge them to explain why the patterns they find work.

How tall is a stack of 20 cups stacked as shown below?



Students can apply this strategy in more advanced levels of BA to answer questions like, "What is the ones digit of 9^{999}?" or, "What is the sum of the first 500 odd numbers?" Try them both by replacing the big numbers with small ones and looking for a pattern. Answers are given below.

Make a Model

Some problems have a lot to keep track of. It's often useful to make a model and act the problem out. It usually only takes a few torn scraps of paper or counters and some trial-and-error to help make sense of the problem. Here's a really tough one to try:

> Iggy, Jan, Kyle, and Lisa are crossing a footbridge at night. Iggy and Jan can each cross in 2 minutes, but Kyle and Lisa both take 3 minutes. The bridge holds at most 2 people. If two people cross together, they walk at the slower person's speed. They have one lantern, and it must be used for every crossing. How many minutes are needed for all four people to reach the other side?

To understand the problem, we can label a paper slip to stand for each person, and use something like a paperclip to represent the lantern. Students can move these back and forth across a "bridge" on their desks, taking notes as they go. This is a problem that also requires organization as you test the possibilities.

Answers to the problems above.

There are 14 small △, 6 medium △, and 2 large △ triangles for a total of 22 triangles. 20 cups reach a height of 23 inches, 9^{999} ends in 9 (like every other odd power of 9), the first 500 odd numbers add up to $500^2 = 250{,}000$, and it takes a minimum of 11 minutes (not 12) for all four people to cross.